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COMBINATORIAL VERSUS DECISION-THEORETIC COMPONENTS OF IMPOSSIBILITY THEOREMS

ABSTRACT. We separate the purely combinatorial component of Arrow's Impossibility Theorem in the theory of collective preference from its decision-theoretic part, and likewise for the closely related result of Blair/Bordes/Kelly/Suzumura. Such a separation provides a particularly elegant proof of the former, via a new 'splitting theorem'.

KEY WORDS: Collective preference, collective choice, impossibility theorems, Arrow.

1. PURPOSE

When reading proofs of impossibility theorems in the theory of collective preference (see, e.g., J. S. Kelly (1978) for an overview), one may be struck by the following impression. On the one hand, the definitional machinery needed for the very formulations, is fairly complex. On the other hand, it is only at certain key points of the proofs that the full machinery is needed, much of the argument appearing to be combinatorial calculation that depends on only a small part of the definitional structure. This suggests the possibility of separating the part belonging essentially to the theory of collective preference from the merely combinatorial part, and formulating the latter without any reference to individual preferences, profiles, or collective rules.

We shall do this for Arrow's (second) impossibility theorem (1963) and for the closely related Blair/Bordes/Kelly/Suzumura theorem (1976). Apart from its intrinsic interest, in helping understand what is involved in those theorems, the decomposition has two further benefits. It permits a particularly elegant proof of Arrow's theorem – roughly speaking, one may follow the essential strategy of the well-known Kirman/Sondermann/Hansson (1972, 1976) proof via ultrafilters, whilst dispensing with the ultrafilters themselves which only complicate the argument. Second, it gives rise to

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a new 'splitting theorem', that generalizes the principal lemma of the Kirman/Sondermann/Hansson proof.

2. BACKGROUND

In principle, we could simply follow the terminology and notation of Kelly (1978). However, Kelly's apparatus is designed to cover such a wide variety of variant impossibility results that it is convenient to streamline the language a little. Readers familiar with the definitions may skip this section, returning to it as needed for memory.

By a *collective preference structure* we mean any quadruple **F** = (N,E,U,F) where:

1. N is a set of cardinality $n \geq 1$ (finite or infinite).
2. E is a set of cardinality ≥ 3 (finite or infinite).
3. U is the set of all n-tuples $u = (\leq_i)_{i \in N}$ where each $\leq_i$ is a relation over E.
4. F is a function taking each $u \in U$ to a relation $\leq$ over E.

We recall that intuitively N, E are understood respectively as sets of *individual agents* and *alternatives* facing them, each relation $\leq_i$ represents the *preferences of individual i* over E, and F is understood as a *rule defining a collective preference relation* $\leq$ out of each *profile* $u = (\leq_i)_{i \in N}$ of all the individual ones.

Note that no constraints are placed here on the individual relations $\leq_i$. It is possible, although rather distracting, to generalize the formulation in this regard, by first fixing, for each $i \in N$, a family $\mathbf{R}_i$ of relations over E that is closed in certain respects, and taking U to be the set of all n-tuples $u = (\leq_i)_{i \in N}$ where each $\leq_i$ is a relation in $\mathbf{R}_i$. We shall be explicit about this kind of generalization after dealing with the simple version. We take the values F(u) to be relations over E, rather than choice functions on subsets of E satisfying suitable conditions; this is merely a matter of presentation.

A collective preference structure **F** = (N,E,U,F) is said to be *finite* iff N is finite. It is said to be *transitive* (resp. *connected*) iff for each $u \in U$ the collective preference relation F(u) alias $\leq$ is so. **F** is said to satisfy the condition of *independence of irrelevant alternatives* (henceforth briefly *independence*) iff whenever $u, u' \in U$ agree on $\{x,y\} \subseteq E$ then F(u) alias $\leq$ and F(u′) alias $\leq'$ agree on $\{x,y\}$. To be precise, iff for all $\{x,y\} \subseteq E$ and $u = (\leq_i)_{i \in N}$, $u' = (\leq_i')_{i \in N}$ in U we have $\leq_i \cap \{x,y\}^2 = \leq_i{}' \cap \{x,y\}^2$ for all $i \in N$ implies $\leq \cap \{x,y\}^2 = \leq' \cap \{x,y\}^2$.

We write $x < y$ to mean that $x \leq y$ but not $y \leq x$. Similarly, for each $i \leq n$, we write $x <_i y$ to mean that $x \leq_i y$ but not $y \leq_i x$. Clearly, even without constraints on $\leq$, $\leq_i$,we have that $<$ and each $<_i$ are asymmetric (and hence also irreflexive). When $S \subseteq N$ we write $x \leq_S y$ (resp $x <_S y$) as shorthand for $x \leq_i y$ for all $i \in S$ (resp. $x <_i y$ for all $i \in S$).

If $\mathbf{F}$ = (N,E,U,F) is a collective preference structure and $x,y \in E$, then a subset $S \subseteq N$ is said to be *decisive for x over y* iff for all $u = (\leq_i)_{i \in N} \in U$, $x <_S y$ implies $x < y$. We note as a limiting case for future use that since each $<_i$ is irreflexive, every non-empty $S \subseteq N$ is decisive for any x over itself. A subset $S \subseteq N$ is said to be *decisive* (for **F**) iff it is decisive for all x over all y in E. We note as another limiting case for future use that since $<$ is asymmetric, the empty set is never decisive. The *weak Pareto condition* (henceforth briefly *Pareto condition)* is the condition that N is decisive. As usual, singleton decisive sets are called *dictators*.

Of all the 'impossibility theorems', the most parsimonious in its hypotheses is the theorem of Sen (1970), commonly known as 'the impossibility of a Paretian liberal', and generalized by Batra and Pattanaik (1972). We recall that this result derives an undesirable property, without hypotheses such as finitude of N, connectivity of the collective preference relation, or independence. It requires only the Pareto condition and the hypothesis (weaker than transitivity of $\leq$) that $<$ is acyclic. The Sen/Batra/Pattanaik theorem tells us the following. *Let* **F** *= (N,E,U,F) be any acyclic collective preference structure satisfying the Pareto condition. Then for any disjoint subsets S, S′ $\subseteq$ N and all x,y,z,w $\in$ E with x $\neq$y, z $\neq$ w, if S is decisive for x over y then S′ fails to be decisive either for z over w or for w over z.* The proof of this result (see, e.g., Kelly (1978) theorems 2.3, 2.5) is so simple and straightforward that there is no combinatorial component to separate from the decision-theoretic argument. On the other hand, in the case of the theorem of Blair, Bordes, Kelly and Suzumura (1976) there is an important combinatorial component that can be separated, as we shall now see.

3. DECOMPOSING THE BLAIR/BORDES/KELLY/SUZUMURA THEOREM

We recall that the theorem derives an undesirable property under the hypotheses of transitivity, independence and Pareto, but without assuming finitude or connectivity, as follows.

BLAIR/BORDES/KELLY/SUZUMURA THEOREM. *Let* $\mathbf{F}$ = *(N,E,U,F) be any transitive collective preference structure satisfying independence and the Pareto condition. Then for all* $S \subseteq N$ *and all distinct* $x,y \in E$, *if S is decisive for x over y then S is decisive.*

Examination of the standard proof of this result (see, e.g., Kelly (1978) lemma 4.1) reveals that it may easily be broken down into the following combinatorial and decision-theoretic components.

COMBINATORIAL LEMMA. *Let E be any set with at least three elements, and let R be a reflexive relation over E such that for all mutually distinct* $x,y,z \in E$, *if* $(x,y) \in R$ *then both* $(x,z) \in R$ *and* $(z,y) \in R$. *Then* $R = I$ *or* $R = E^2$.

DECISION-THEORETIC LEMMA. *Let* $\mathbf{F}$ = *(N,E,U,F) be any transitive collective preference structure satisfying independence and the Pareto condition. Then for all* $S \subseteq N$ *and all mutually distinct* $x,y,z \in E$, *if S is decisive for x over y then it is decisive for both x over z and z over y.*

Clearly, the theorem follows from the two lemmas by defining R in the combinatorial one by the rule $(x,y) \in R$ iff S is decisive for x over y. The proof of the decision-theoretic lemma is standard, and may be found in, e.g., Kelly (1987) lemma 4.1, from which it is also not difficult to abstract a proof for the combinatorial lemma, as follows.

Proof of the Combinatorial Lemma. Assume the hypotheses of the lemma and suppose $R \neq I$. Let $a, b \in E$; we want to show that $(a,b) \in R$. In the case $a = b$ we are done by reflexivity of R, so suppose $a \neq b$. Since $R \neq I$ there are $x, y \in E$ with $x \neq y$ and $(x,y) \in R$.

Case 1: Suppose $b \neq x$. If $b = y$ then since $(x,y) \in R$ we have immediately $(x,b) \in R$, whilst if on the other hand $b \neq y$ then x, y, b are mutually distinct so since $(x,y) \in R$ we also have by hypothesis that $(x,b) \in R$. Thus in either subcase, $(x,b) \in R$. Hence if $a = x$ we have immediately $(a,b) \in R$, whilst if on the other hand $a \neq x$ then x, a, b are mutually distinct so since $(x,b) \in R$ we also have by hypothesis that $(a,b) \in R$. Thus in either subcase, $(a,b) \in R$.

Case 2: Suppose $b = x$. If on the one hand $a \neq y$ then a, b, y are mutually distinct so since $(b,y) = (x,y) \in R$ we have by the hypotheses that $(a,y) \in R$ so again $(a,b) \in R$. If on the other hand $a = y$ we have $(b,a) = (x,y) \in R$. Since E has at least three elements there is a $z \in E$ with a, b, z mutually distinct. Hence by the hypotheses since $(b,a) \in R$ we have $(b,z) \in R$ so again $(a,z) \in R$ and again $(a,b) \in R$.

4. DECOMPOSING ARROW'S THEOREM

In the case of Arrow's (second) impossibility theorem, the separation of combinatorial from set-theoretic ingredients is rather more subtle, but also more rewarding. We recall the theorem itself.

ARROW'S IMPOSSIBILITY THEOREM. *Let* $\mathbf{F} = (N,E,U,F)$ *be any finite transitive, connected collective preference structure satisfying independence and the Pareto condition. Then N contains a dictator.*

We obtain this as a direct consequence of the following 'splitting theorem', generalizing a lemma of Kirman/ Sondermann/Hansson (who in effect consider the case $D = N$).

SPLITTING THEOREM. *Let* $\mathbf{F} = (N,E,U,F)$ *be any transitive, connected collective preference structure satisfying independence. Whenever* $D \subseteq N$ *is decisive for* $\mathbf{F}$ *then for all* $S \subseteq D$ *either S or D-S is decisive for* $\mathbf{F}$.

Proof of Arrow's Theorem from the Splitting Theorem. By the Pareto condition, N is decisive. Since N is finite, it has a minimal decisive subset D. Since the empty set is not decisive, D has at least one element. But by the splitting theorem, D cannot have more than one element, and we are done.

We now separate the combinatorial and decision-theoretic parts of the splitting theorem, and prove them separately.

COMBINATORIAL LEMMA. *Let E be any set with at least three elements, and let R_1, R_2 be reflexive relations over E such that:*

(1) R_1 is almost disjoint with the converse of R_2, in the sense that $R_1 \cap R_2^{-1} = I$, where I is the identity relation over E,

(2) For all mutually distinct $x,y,z \in E$, if $(x,y) \notin R_i$ then $(y,z) \in R_j$ $(i \neq j)$.

Then either $R_1 = E^2$ or $R_2 = E^2$.

DECISION-THEORETIC LEMMA. *Let* $\mathbf{F} = (N,E,U,F)$ *be any transitive, connected collective preference structure satisfying independence. Let D be a decisive subset of N and let S be any non-empty subset of D. Then for all mutually distinct $x, y, z \in E$, if S is not decisive for x over y then D-S is decisive for y over z.*

Proof of Splitting Theorem from the Two Lemmas. Let D be a decisive subset of N, and let $S \subseteq D$. If $S = \emptyset$ or D-S $= \emptyset$ then we are done, so suppose $S \neq \emptyset$ and D-S $\neq \emptyset$. Consider the relations R_1, R_2 over E defined by putting $(x,y) \in R_1$ (resp. R_2) iff D-S (resp. S) is decisive for x over y. We have already noted as a limiting case, that every non-empy subset of N is decisive for each $x \in E$ over itself, i.e. the relations R_1, R_2 are both reflexive. To show that R_1 and R_2^{-1} are almost disjoint, let x,y be distinct elements of E and suppose $(x,y) \in R_1$. We need to check that $(y,x) \notin R_2$. Since x and y are distinct and S, D-S are disjoint, there is clearly a profile $u = (\leq_i)_{i \in N}$ with $x <_i y$ for all $i \in$ D-S and $y <_i x$ for all $i \in S$. Then since $(x,y) \in R_1$ we have $x < y$ so by asymmetry not $y < x$ and thus $(y,x) \notin R_2$. Thus R_1, R_2 satisfy condition (1) of the combinatorial lemma. The decision-theoretic lemma tells us that they also satisfy condition (2). The combinatorial lemma then tells us that either $R_1 = E^2$ or $R_2 = E^2$, i.e. either D-S or S is decisive.

Proof of the Combinatorial Lemma. In fact this follows as a simple application of the earlier combinatorial lemma. For suppose the hypotheses of the lemma to be proven. First, we observe that for i = 1,2, R_i = I or $R_i = E^2$. For let x, y, z be any mutually distinct elements of E and suppose $(x,y) \in R_i$. By hypothesis (1) we have $(y,x) \notin R_j$, and so by hypothesis (2) applied directly we get $(x,z) \in R_i$, and applied in contraposed form also $(z,y) \in R_i$.. Hence by the earlier combinatorial lemma, for i = 1,2, R_i = I or $R_i = E^2$. Since E has at least three mutually distinct elements, hypothesis (2) also implies that we cannot have $R_1 = I = R_2$. Hence either $R_1 = E^2$ or $R_2 = E^2$ as desired.

Proof of the Decision-Theoretic Lemma. Suppose the hypotheses of the lemma, let x,y,z be distinct elements of E, and suppose that S is not decisive for x over y; we need to show that D-S is decisive for y over z.

Let $u = (\leq_i)_{i \in N}$ be any profile with $y <_{D-S} z$. We need to show that y < z. Since S is not decisive for x over y there is a profile $u' = (\leq_i')_{i \in N}$ with $x <_S' y$ but not $x <' y$. Now consider any profile $u'' = (\leq_i'')_{i \in N}$ such that.

For all i ∈ D-S, $\leq_i''$ agrees with $\leq_i'$ on {x,y}, and puts both x, y $<_i''$ z

For all i ∈ S, $\leq_i''$ agrees with $\leq_i$ on {y,z}, and puts both $x <_i'' y$, z

For all i ∈ N-D, $\leq_i''$ agrees with $\leq_i'$ on {x,y} and with $\leq_i$ on {y,z}.

Clearly, since x,y,z are mutually distinct, such a u″ exists. It is helpful to sum up this information in the following table, where e.g. the < in the column for u″, subcolumn for {x,y}, row for S, means that $x <_S'' y$, and where a,b,c,d are unknown. In the bottom row, $\not<$ in the leftmost cell means that we have $x \not< y$ when $\leq$ is chosen to be F(u′).

We can now deduce values for the four remaining empty squares in the bottom row. From left to right they are $\not<$, <, <, <. The verification is as follows.

First, for each i ∈ N, $\leq_i''$ agrees with $\leq_i'$ on {x,y}, so by independence $\leq''$ agrees with $\leq'$ on {x,y}, and thus $x \not<'' y$. Thus also by connectivity of $\leq''$ we have $y \leq'' x$. Next, for each i ∈ D, $x <_i'' z$ so since D is decisive, $x <'' z$. Putting these together we get by transi-

TABLE I

	u′	u″			u
	{x,y}	{x,y}	{x,z}	{y,z}	{y,z}
D-S	a	a	<	<	<
S	<	<	<	b	b
N-D	c	c		d	d
F(.)	$\not<$				

tivity of $\leq''$ that y $<''$ z. Finally, for each i $\in$ N, $\leq_i''$ agrees with $\leq_i$ on {y,z}, so that by independence $\leq''$ agrees with $\leq$ on {y,z} and thus y $<$ z as desired.

5. COMMENTS ON THE PROOF

1. *Use of key hypotheses*: In the above proof of Arrow's theorem, all applications of independence, transitivity and connectivity of the collective preference relation, are concentrated at a single point (the last paragraph of the proof of the decision-theoretic lemma). The finitude of N and the Pareto condition are not used there, but in the extraction of Arrow's theorem from the splitting theorem .

2. *Adding constraints on individual preferences*: In the definition of a collective preference structure, we have allowed individual preferences to be arbitrary relations $\leq_i$ over E. It is known that Arrow's theorem continues to hold when the individual preferences are constrained to be well-behaved in various ways, notably transitive, connected, or antisymmetric. Inspection of our proof reveals that it continues to go through without any change under such constraints. Indeed, more generally, it goes through without change when for each individual i $\in$ N we allow i's preference relation to be drawn from any family $\mathbf{R}_i$ of relations over E that is closed in four respects. The first of these, which suffices to guarantee the existence of the profile u defined in the proof of the splitting theorem, is: *for all distinct $x, y \in E$ there is a $\leq_i$ in $\mathbf{R}_i$ with $x <_i y$.* The remaining three, which together clearly suffice to guarantee the existence of the

profile u″ defined in the proof of the decision-theoretic lemma, are: *for all* $\leq_i$, $\leq_i'$ *in* $\mathbf{R}_i$ *and all mutually distinct x, y, z* $\in$ *E:*

there is a $\leq_i''$ in $\mathbf{R}_i$ that agrees with $\leq_i'$ on {x,y}, and puts both x, y $<_i''$ z,
there is a $\leq_i''$ in $\mathbf{R}_i$ that agrees with $\leq_i$ on {y,z}, and puts both x $<_i''$y, z ,
there is a $\leq_i''$ in $\mathbf{R}_i$ that agrees with $\leq_i'$ on {x,y} and with $\leq_i$ on {y,z}.

It is easy to check that for any given subset of {transitivity, connectivity, antisymmetry}, if $\mathbf{R}_i$ is chosen to be the set of all relations over E that satisfy the conditions in that subset, then it is closed in the above four respects.

Such constraints on the sets $\mathbf{R}_i$ leave U as a product $\prod\{\mathbf{R}_i : i \in N\}$. We do not here consider possible further generalizations in that respect.

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